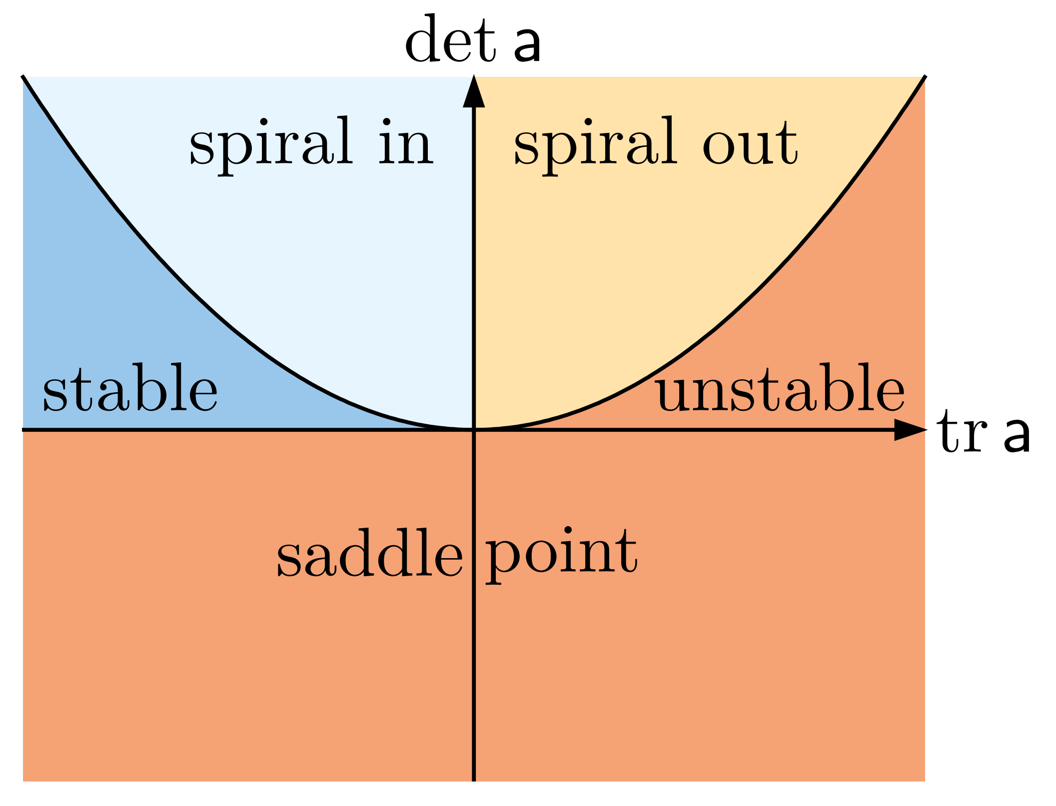
Chaos – Complexity – Evolution

3. Nonlinear Dynamical Systems (II)

- two-dimensional autonomous systems
  - characterization
  - linear stability analysis
  - invariant sets and manifolds



Kurt Roth-Fauchère, 2021

two-dimensional autonomous system

- development equation

$$\begin{aligned}\dot{u}_1 &= f_1(u_1, u_2)\\ \dot{u}_2 &= f_2(u_1, u_2)\end{aligned}$$

$$\dot{\mathbf{u}} = \mathbf{f}(\mathbf{u})$$

- flow $\quad \dot{\mathbf{u}}(\mathbf{u})\ , \quad \mathbf{f}(\mathbf{u})$

- nullclines

$$f_i(\mathbf{u}) = 0\ , \quad \dot{u}_i = 0$$

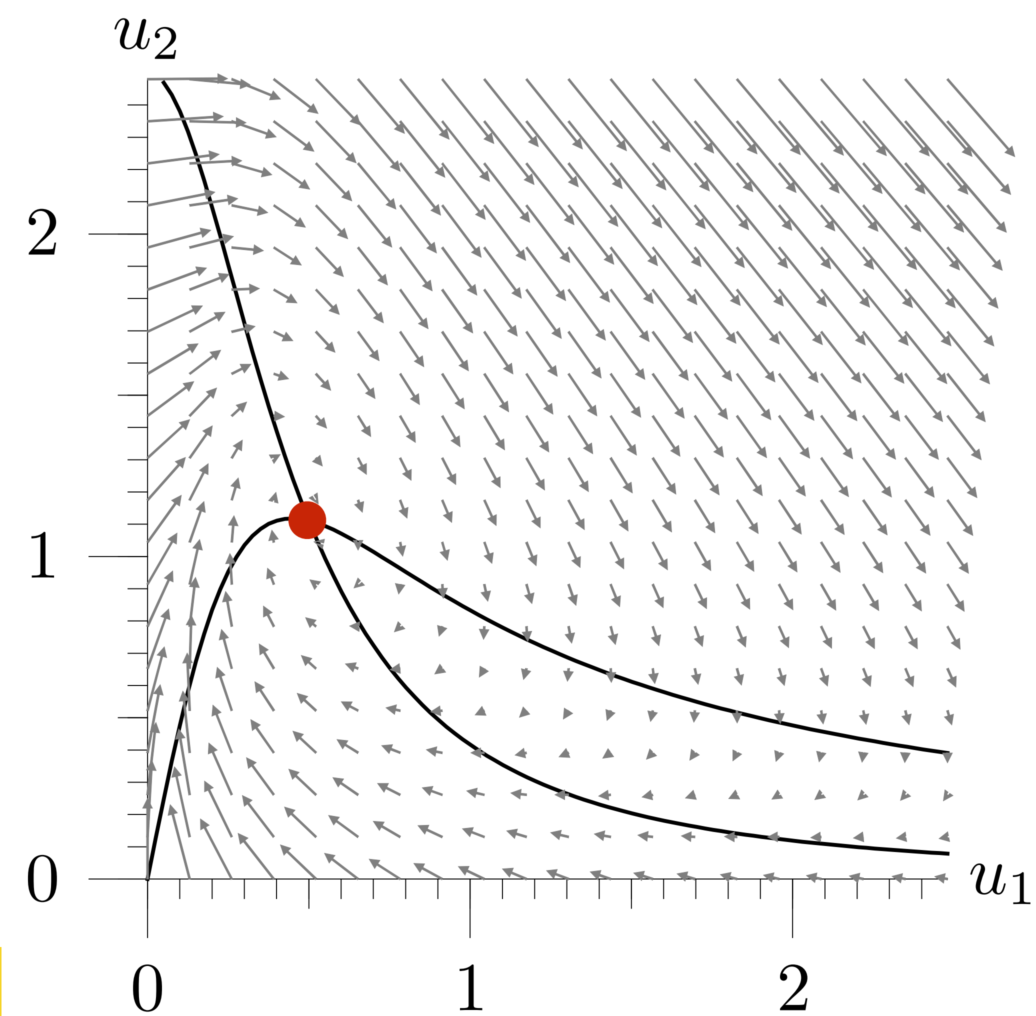
nullclines separate state space into regions, **but** trajectories can cross them

- intersections of nullclines are fixpoints

- trajectory

$$\mathbf{u}(t) = \mathbf{u}_0 + \int_0^t \dot{\mathbf{u}}\big(\mathbf{u}(\tau)\big)\, \mathrm{d}\tau$$

discretize



Euler forward scheme

$$\mathbf{u}(t_n) \approx \mathbf{u}_0 + \sum_{i=1}^{n} \dot{\mathbf{u}}\big(\mathbf{u}(t_i)\big)\Delta t\ , \quad t_i = i\Delta t$$

(see Appendix for higher order scheme – Runge-Kutta Cash-Karp – that is much more efficient)

example – glycolysis model

[Strogatz, 1994]

production of u_1 through autocatalytic consumption of u_2

generic for autocatalytic process

actual glycolysis is much more complicated

constant/outflow

$$\dot{u}_1 = -u_1 + u_2[\alpha + u_1^2]$$

spontaneous decay into u_1
(no autocatalysis without a direct process)

autocatalytic decomposition by u_1

$$\dot{u}_2 = \beta - u_2[\alpha + u_1^2]$$

constant feed
(driver)

$$\alpha > 0 \,, \quad \beta > 0$$

- expected phenomenology
- verify: choose parameters, draw trajectories $u_1(t)$ & $u_2(t)$ (see exercises)

the model in words

general structure of a sustained system (recall cows feeding on grass)

• u_2 constantly fed with rate β	inflow
• u_2 decays spontaneously into u_1 with rate α… • …and is decomposed by u_1 in autocatalytic process	do something
• u_1 decays with rate 1 (sets unit of time)	outflow

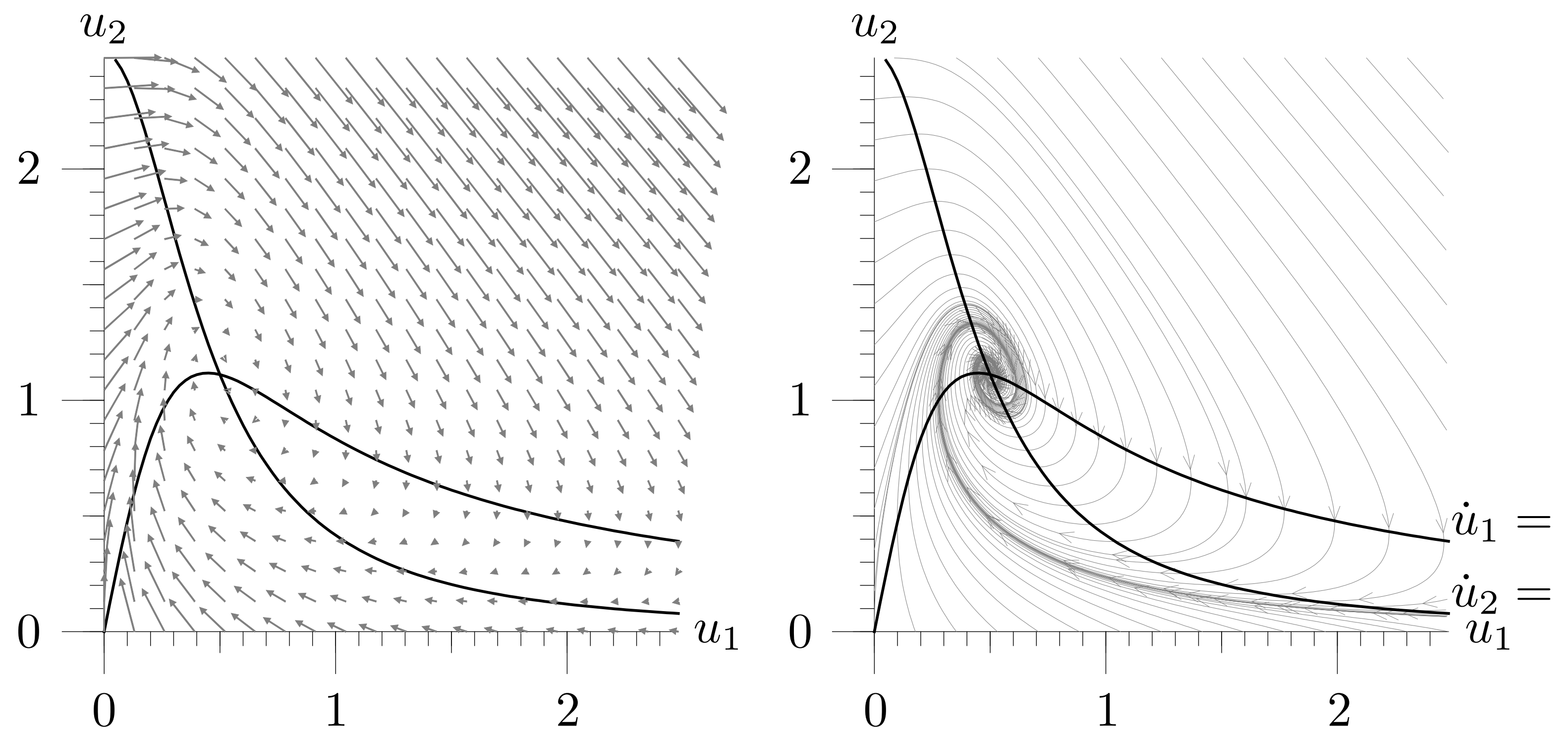
example – glycolysis model

[Strogatz, 1994]

$$\dot{u}_1 = -u_1 + u_2[\alpha + u_1^2]$$

$$\dot{u}_2 = \beta - u_2[\alpha + u_1^2]$$

$$\alpha > 0\ , \quad \beta > 0$$



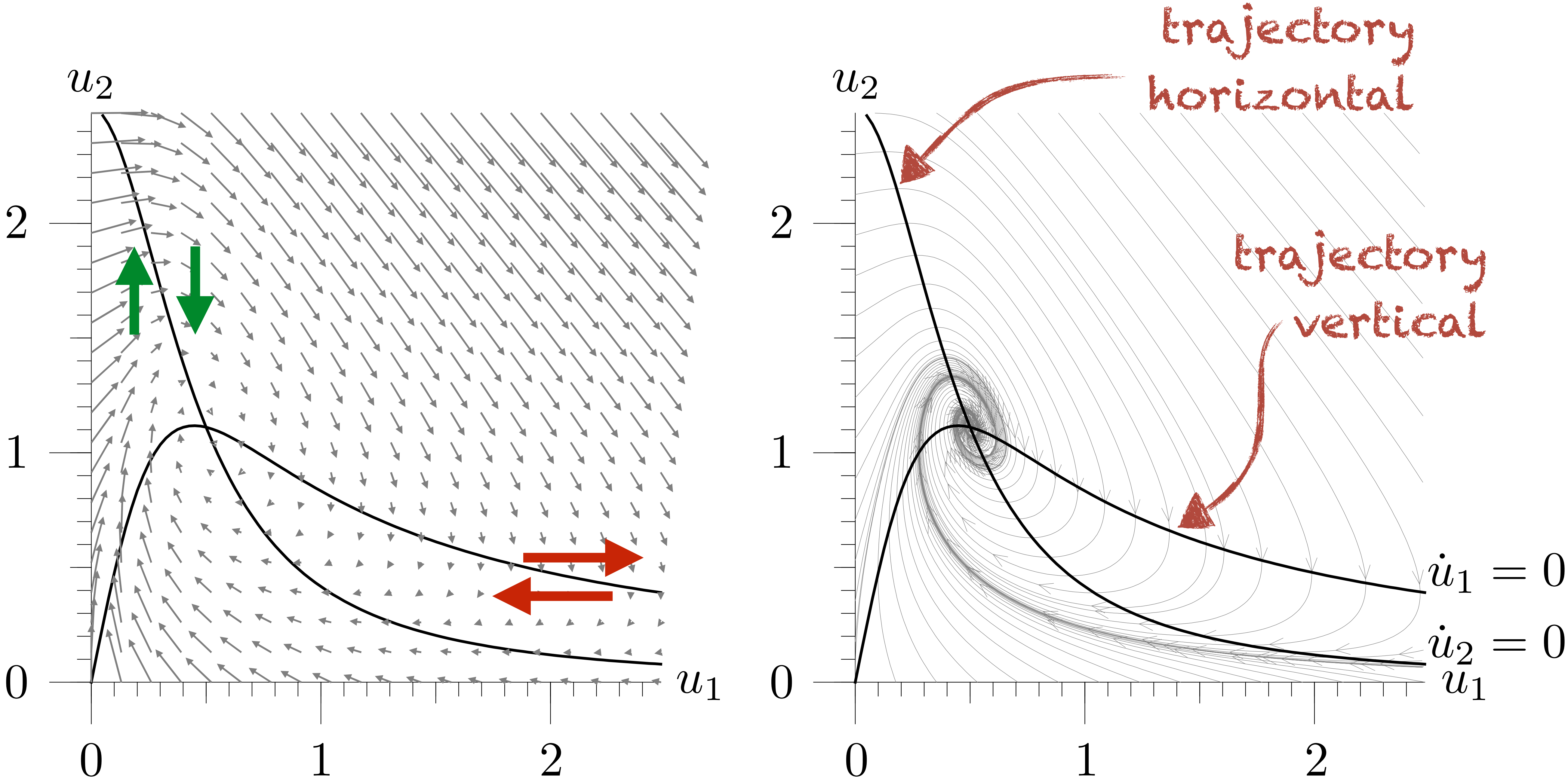
nullclines: $\dot{u}_1 = 0 \quad \rightarrow \quad u_{2_1} = \dfrac{u_1}{\alpha + u_1^2}$

$\dot{u}_2 = 0 \quad \rightarrow \quad u_{2_2} = \dfrac{\beta}{\alpha + u_1^2}$

fixpoint: $(u_{0_1}, u_{0_2}) = \left(\beta, \dfrac{\beta}{\alpha + \beta^2}\right)$

example – glycolysis model

[Strogatz, 1994]



-> graphical solution (in analogy to 1d)

linear stability analysis…

in analogy to 1d case

- development of $\mathbf{u} = \mathbf{u}_0 + \boldsymbol{\varepsilon}$ with $\mathbf{f}$ continuously differentiable at $\mathbf{u}_0$ ($\boldsymbol{\varepsilon}$: perturbation, $\mathbf{u}_0$: fixpoint)

$$\Rightarrow \quad \dot{\mathbf{u}} = \dot{\mathbf{u}}_0 + \dot{\boldsymbol{\varepsilon}} = \mathbf{f}(\mathbf{u}_0 + \boldsymbol{\varepsilon}) = \mathbf{f}(\mathbf{u}_0) + \mathsf{a}\boldsymbol{\varepsilon} + \mathcal{O}(\|\boldsymbol{\varepsilon}\|^2), \quad \mathsf{a}\text{: Jacobian matrix}$$

- linear approximation and $\mathbf{u}_0$ fixpoint ($\dot{\mathbf{u}}_0 = 0 = \mathbf{f}(\mathbf{u}_0)$)

$$\dot{\boldsymbol{\varepsilon}} = \mathsf{a}\boldsymbol{\varepsilon}\ , \quad \mathsf{a} = \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nn} \end{pmatrix}\ , \quad a_{ij} = \frac{\partial f_i}{\partial u_j}\Big|_{\mathbf{u}=\mathbf{u}_0}$$

a: square matrix, if diagonalizable $\mathsf{a} = \mathsf{P}\,\mathsf{D}\,\mathsf{P}^{-1}$

- P: matrix with eigenvectors of a as columns
- D: diagonal matrix with corresponding eigenvalues (possibly complex)
- not diagonalizable: defective with incomplete basis (sum of diagonalizable and nilpotent matrix: Jordan normal form)

...linear stability analysis

- eigen decomposition: $\mathsf{a}\mathbf{v} = \sigma\mathbf{v}$, $\mathbf{v}$: eigenvector with eigenvalue σ

  $\hookrightarrow \det\left[\mathsf{a} - \sigma\mathsf{I}\right] = 0 \quad \rightarrow$ characteristic polynomial with roots $\{\sigma_1, \ldots, \sigma_n\}$, some of which conjugate complex

- express $\dot{\boldsymbol{\varepsilon}} = \mathsf{a}\boldsymbol{\varepsilon}$ in eigenbasis $\{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ of a to obtain n decoupled linear ODEs (retain symbol ε for notational simplicity)

  $\dot{\varepsilon}_i = \sigma_i \varepsilon_i \quad \rightarrow \quad \varepsilon_i(t) = \varepsilon_{i_0} \exp(\sigma_i t)$ in linear approximation

  development of deviation ε_i in direction of eigenvector $\mathbf{v}_i$ with associated eigenvalue σ_i (conj. complex: go to polar coordinates)

- linear stability:
  - $\mathbf{u}_0$ stable if $\mathrm{Re}(\sigma_i) < 0$, $\forall i$
  - $\mathbf{u}_0$ unstable if $\mathrm{Re}(\sigma_i) > 0$ for any i

  names hyperbolic point: $\mathrm{Re}(\sigma_i) \neq 0$, $\forall i$

  hyperbolic saddle point: hyperbolic & different signs

  critical (or central) point: $\mathrm{Re}(\sigma_i) = 0$
  (fixpoint in direction $\mathbf{v}_i$; a non-diagonalizable)

invariant sets and manifolds

invariant sets and manifolds

extend the notion of invariant point (fixpoints)

propagator $\mathcal{D}_t\,\mathbf{u}(0) := \mathbf{u}(0) + \int_0^t \mathbf{f}(\mathbf{u}(\tau))\,\mathrm{d}\tau = \mathbf{u}(t)$

invariant set set $\mathfrak{S}$ invariant $\Leftrightarrow$ $\mathcal{D}_t\,\mathbf{u} \in \mathfrak{S}$, $\forall \mathbf{u} \in \mathfrak{S}$ and $\forall t$

how to draw those manifolds?

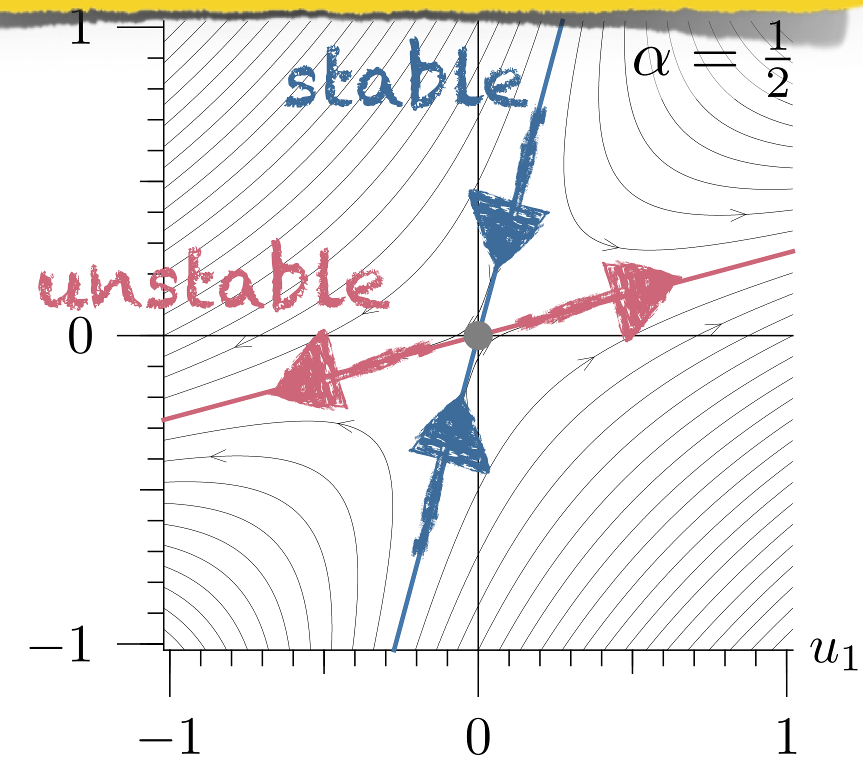
- unstable: start near to fixpoint
- stable: invert time (sign of f(u))
  do as with unstable

every trajectory is an invariant manifold

consider trajectories associated with fixpoints

stable vs unstable manifold – example

$$
\begin{aligned}
\dot{u}_1 &= u_1 - \alpha u_2 \\
\dot{u}_2 &= \alpha u_1 - u_2
\end{aligned}
$$



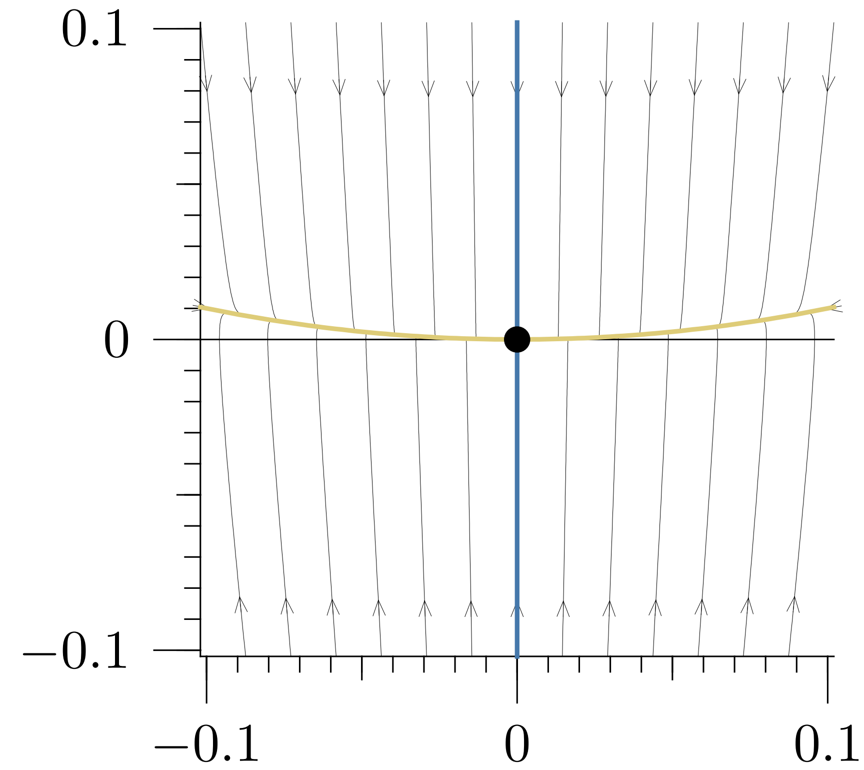
eigensystem

$$
\mathbf{v}_\pm = \begin{pmatrix} \frac{1}{\alpha}[1 \pm \gamma] \\ 1 \end{pmatrix}, \quad \sigma_\pm = \pm\gamma\,, \quad \gamma = \sqrt{1-\alpha^2}
$$

invariant manifolds

center manifold – example

$$\begin{aligned}\dot{u}_1 &= -u_1 u_2 \\ \dot{u}_2 &= -u_2 + u_1^2 - 2u_2^2 \ ,\end{aligned}$$



at fixpoint (0,0):

$$\mathbf{v}_s = \begin{pmatrix} 0 \\ 1 \end{pmatrix},\ \mathbf{v}_c = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \qquad \sigma_{s,c} = \{-1, 0\}$$

$$\begin{aligned}\dot{u}_1^{\text{lin}} &= 0 \\ \dot{u}_2^{\text{lin}} &= -u_2 \ ,\end{aligned} \quad \xrightarrow{\text{fast dynamics}} \quad u_2(t) = u_0 \exp(-t)$$

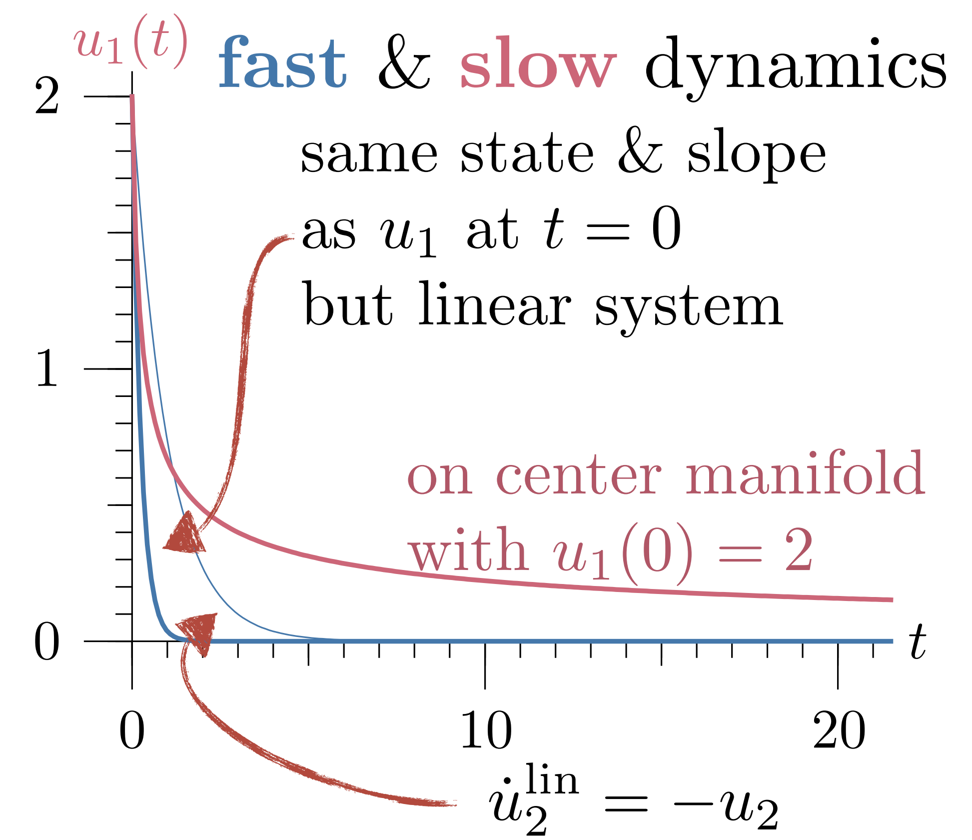
better approximation of center manifold

$$\dot{u}_2 = 0 \ \longrightarrow \ u_2(u_1) = \frac{1}{4}\left[-1 + \sqrt{1 + 8u_1^2}\right]$$

$$u_2(u_1) \approx u_1^2 \quad \text{(Taylor)}$$

insert into first – $\dot{u}_1 = -u_1^3$ – and integrate

$$u_1(t) = \left[2t + \frac{1}{u_0^2}\right]^{-\frac{1}{2}}$$



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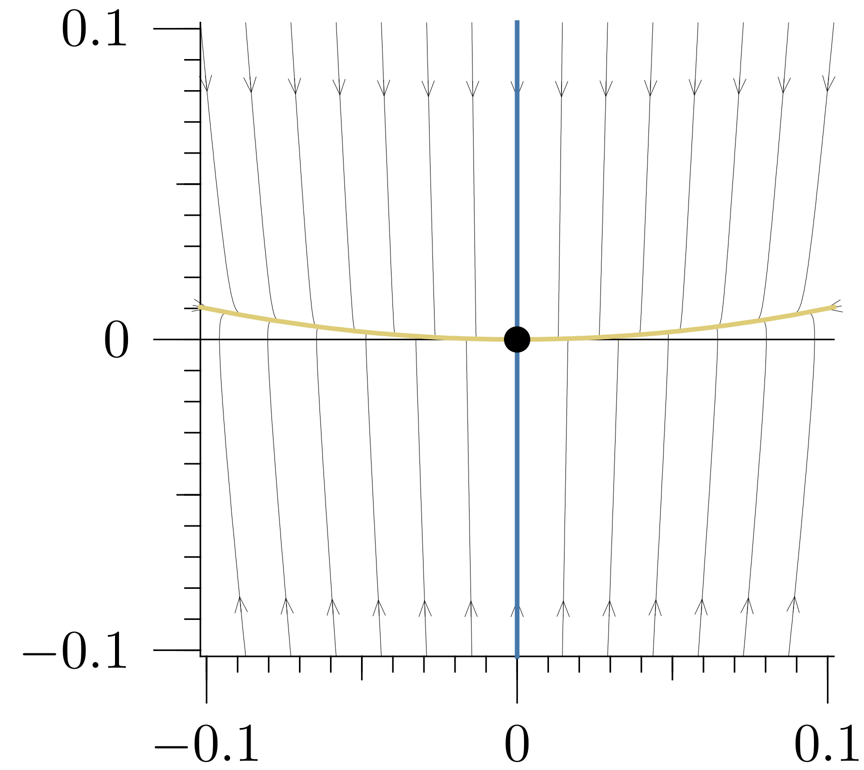

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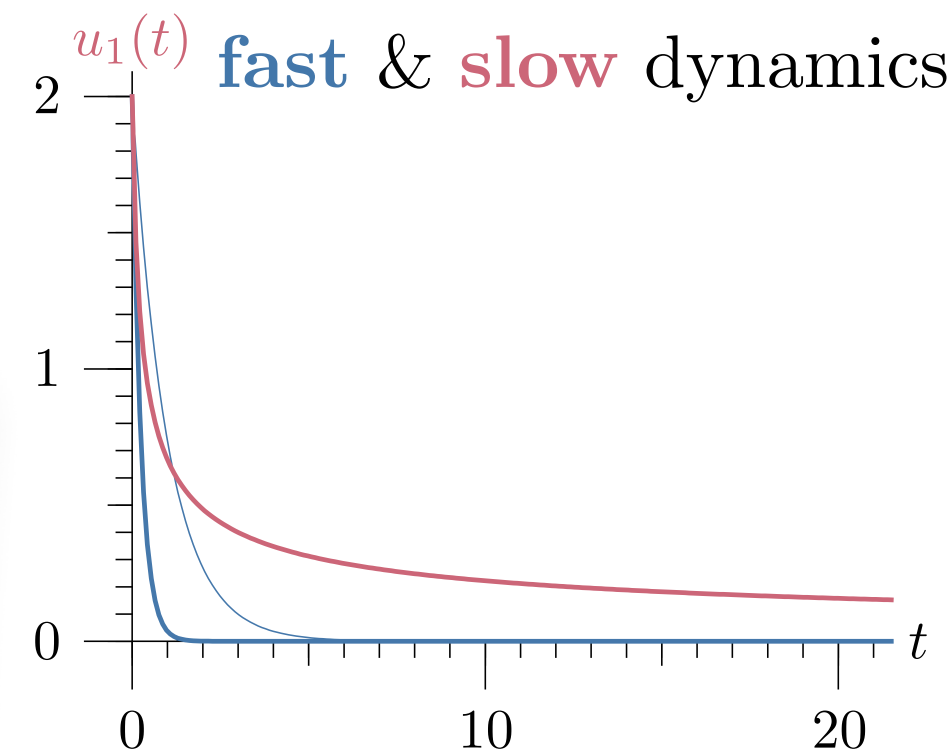

separation of time scales:
- fast dynamics forces system to low-dimensional center manifold
- slow dynamics determines long-time phenomenology (enslaves fast variables)

invariant manifolds

center manifold – example

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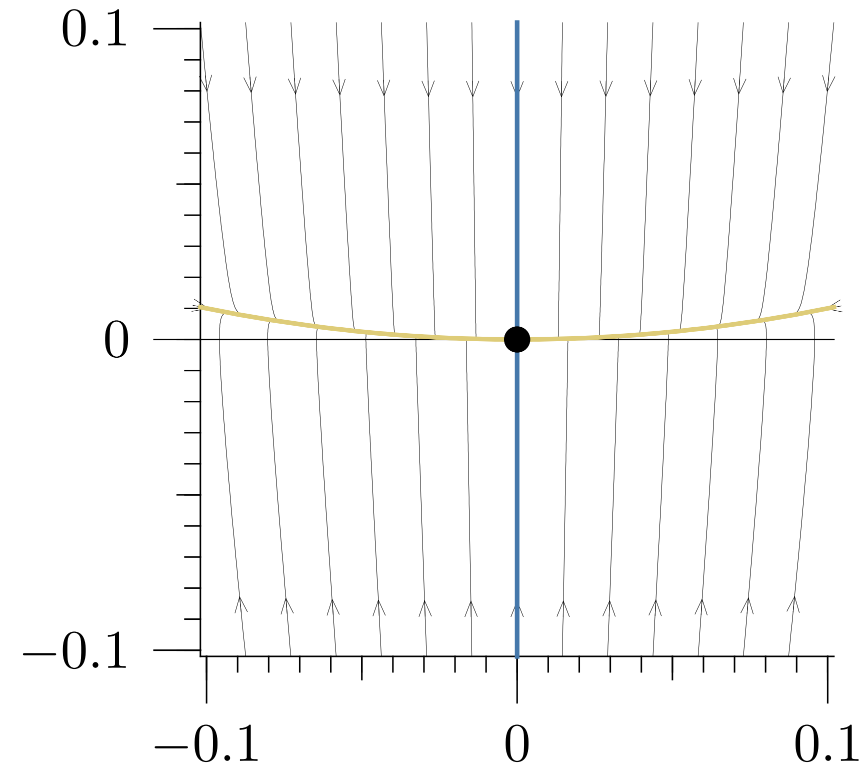

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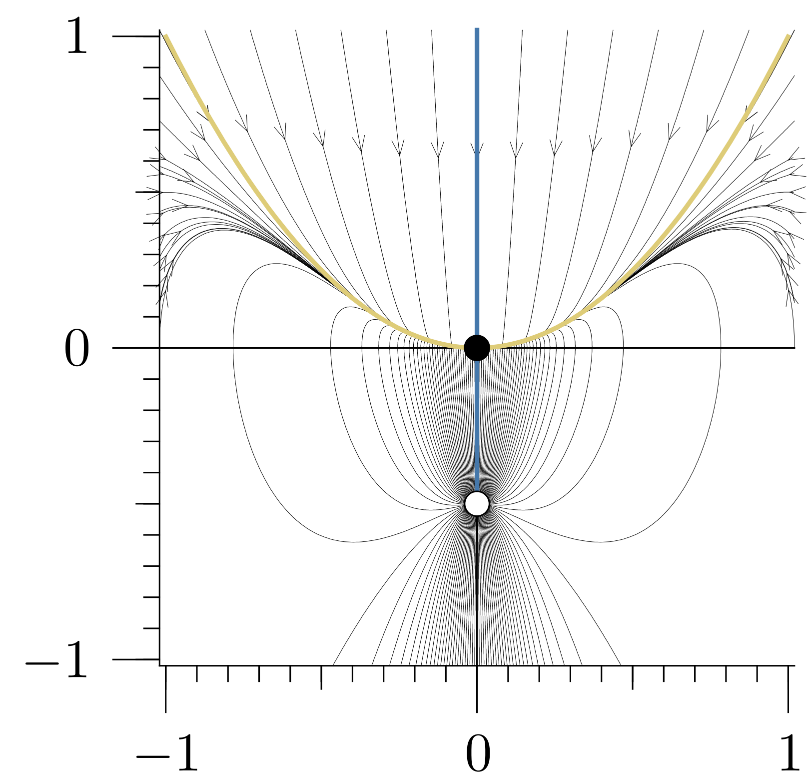

separation of time scales:

- fast dynamics forces system to low-dimensional center manifold
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more on linear stability analysis

Lyapunov exponents

- recall development of small perturbation $\boldsymbol{\varepsilon}_0$ of fixpoint $\mathbf{u}_0$ in eigenspace of a

$$\boldsymbol{\varepsilon}(t) = \sum_i \varepsilon_{i_0} \exp(\sigma_i t)\mathbf{v}_i$$

- can use eigen decomposition at any state $\mathbf{u}$, need not be a fixpoint

$\hookrightarrow$ development of small separation $\Delta\boldsymbol{\varepsilon}$ between two states $\mathbf{u}_A$ and $\mathbf{u}_B$

$$\Delta\boldsymbol{\varepsilon}(t) = \sum_i \Delta\varepsilon_{i_0} \exp(\sigma_i t)\mathbf{v}_i$$

σ_i are often called *Lyapunov exponents*

- use σ_i to define **deterministic time horizon**

$$\tau_{d_i} = \frac{1}{\sigma_i}\ , \quad \text{for } \sigma_i > 0$$

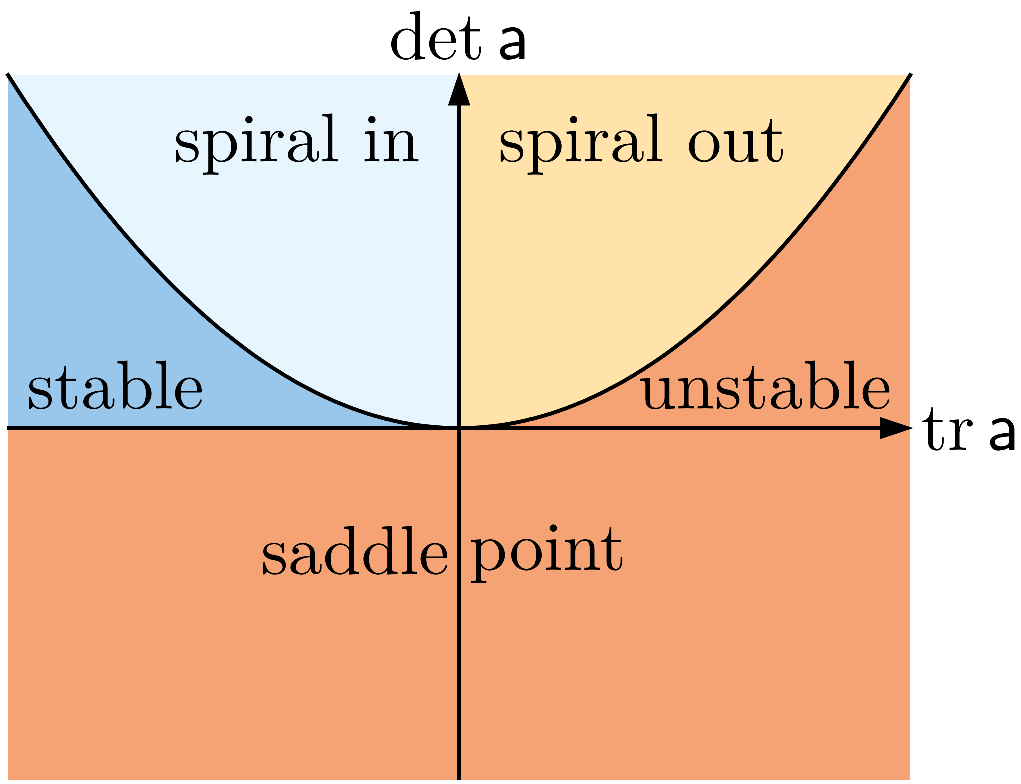
Linearization Theorem (Hartman-Grobman)

is linearization "safe"?

In a neighborhood of the hyperbolic saddle point $\mathbf{u}_0$,
the nonlinear system and its linearization are topologically equivalent.

linear stability – two-dimensional system

aim: express stability in terms of matrix invariants



n-dimensions:
coefficients of characteristic polynomial may be expressed in terms of determinants of matrices with entries tr[$\mathbf{a}^k$]
(trace of powers of **a**)

2d: characteristic polynomial

$$\det[\mathsf{a} - \sigma \mathsf{I}] = \sigma^2 - [\mathrm{tr}\, \mathsf{a}]\sigma + \det \mathsf{a} = 0$$

$$\begin{aligned} \mathrm{tr}\ \mathsf{a} &= a_{11} + a_{22} \\ \det \mathsf{a} &= a_{11}a_{22} - a_{12}a_{21} \end{aligned}$$

$$\sigma^{\pm} = \frac{1}{2}\mathrm{tr}\ \mathsf{a} \pm \frac{1}{2}\sqrt{[\mathrm{tr}\ \mathsf{a}]^2 - 4\det \mathsf{a}}$$

- $\mathrm{Re}(\sigma) < 0$: stable
- $\mathrm{Re}(\sigma) > 0$: unstable
- $\mathrm{Im}(\sigma) = 0$: monotonic
- $\mathrm{Im}(\sigma) \neq 0$: spiraling

linear stability – two-dimensional system

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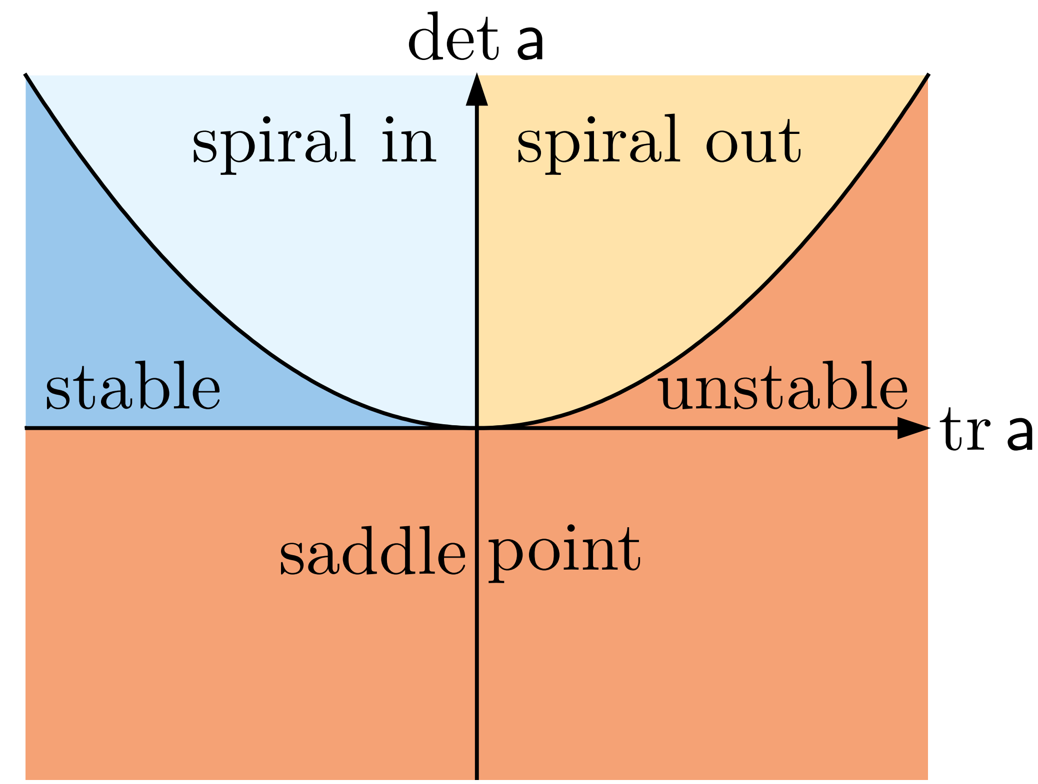


$$\sigma^{\pm} = \frac{1}{2}\text{tr } \mathsf{a} \pm \frac{1}{2}\sqrt{[\text{tr } \mathsf{a}]^2 - 4\det \mathsf{a}}$$

to become operationally useful,
translate into parameter space of specific system

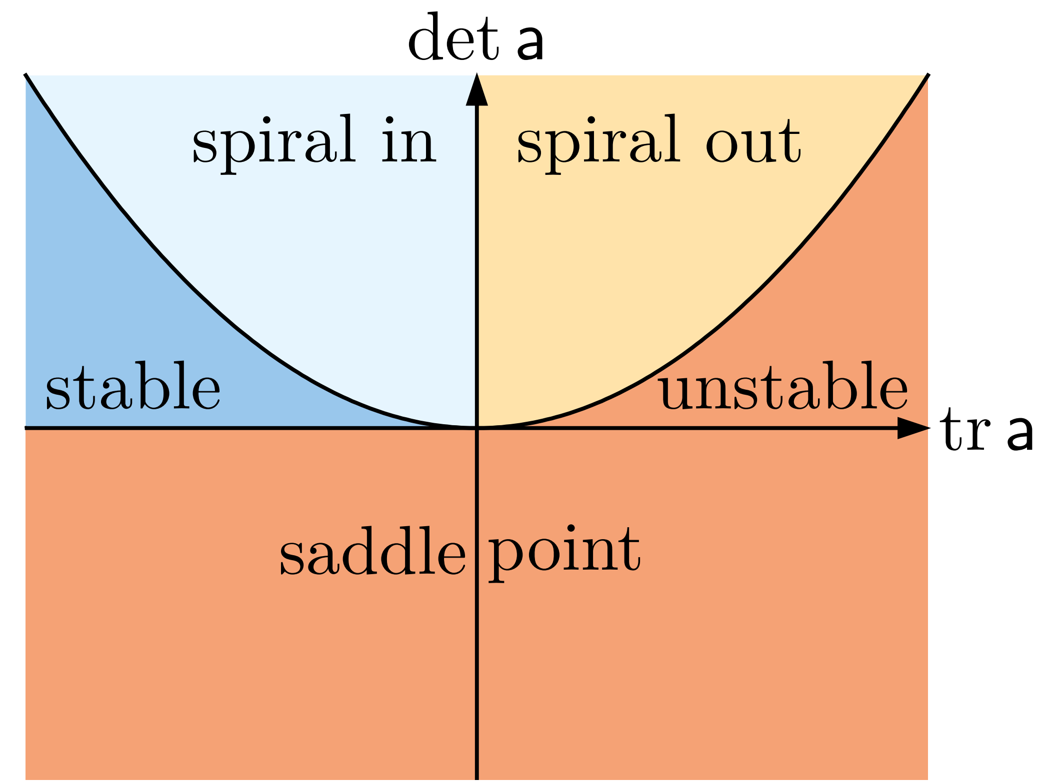
overall three-step procedure:

1. calculate fixpoint $\mathbf{u}_0$ by setting $\dot{\mathbf{u}} = 0$
2. calculate Jacobian matrix $\partial f_i / \partial u_j$ and evaluate it at $\mathbf{u}_0$: $\rightarrow \mathsf{a}$
3. calculate $\det \mathsf{a}$ and $\text{tr}\, \mathsf{a}$, now expressed in terms of the system parameters, and identify in parameter space the regions where $\det \mathsf{a} \gtrless 0$ and $\text{tr}\, \mathsf{a} \gtrless 0$

linear stability – two-dimensional system

example – glycolysis model

$$\dot{u}_1 = f_1(u_1,u_2) = -u_1 + u_2[\alpha + u_1^2]$$
$$\dot{u}_2 = f_2(u_1,u_2) = \beta - u_2[\alpha + u_1^2]$$



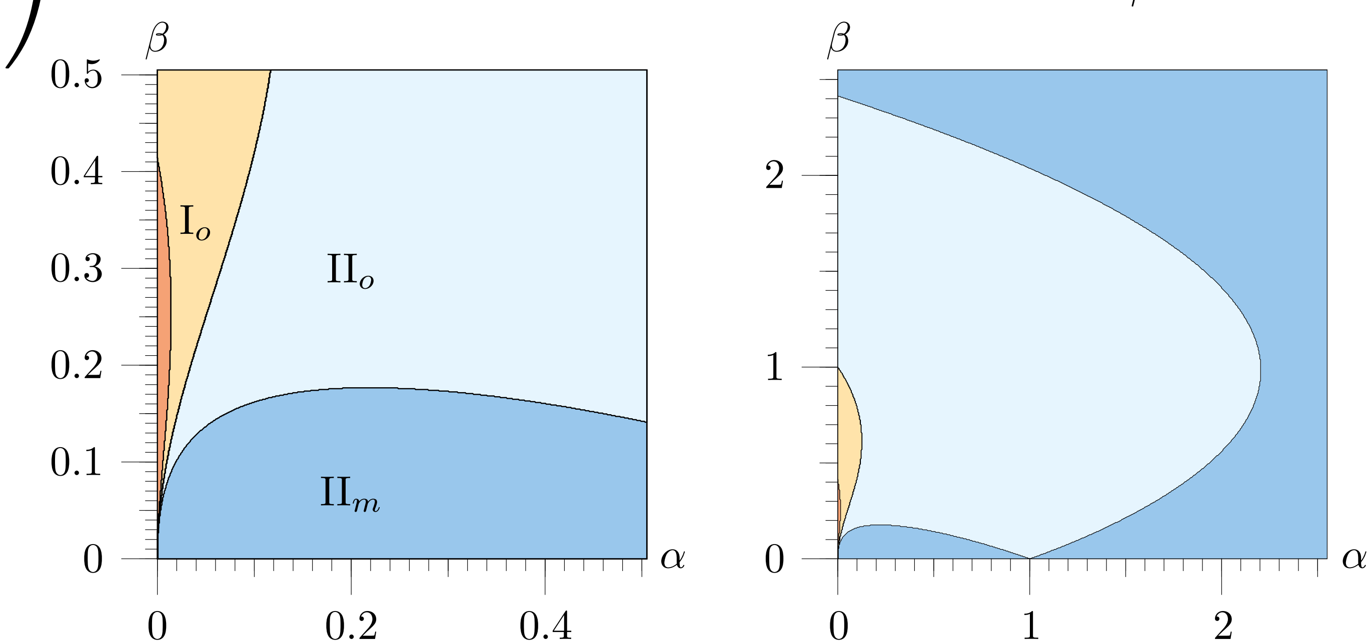
Jacobian matrix

$$\mathsf{a} = \begin{pmatrix} -1 + 2u_1u_2 & \alpha + u_1^2 \\ -2u_1u_2 & -\alpha - u_1^2 \end{pmatrix}, \quad a_{ij} = \frac{\partial f_i}{\partial u_j}$$

evaluate at fixpoint

$$\mathsf{a}|_{\mathbf{u}=\mathbf{u}_0} = \begin{pmatrix} -1 + 2\frac{\beta^2}{\alpha+\beta^2} & \alpha + \beta^2 \\ -2\frac{\beta^2}{\alpha+\beta^2} & -\alpha - \beta^2 \end{pmatrix}$$

recall fixpoint: $(u_{0_1}, u_{0_2}) = \left(\beta, \dfrac{\beta}{\alpha + \beta^2}\right)$

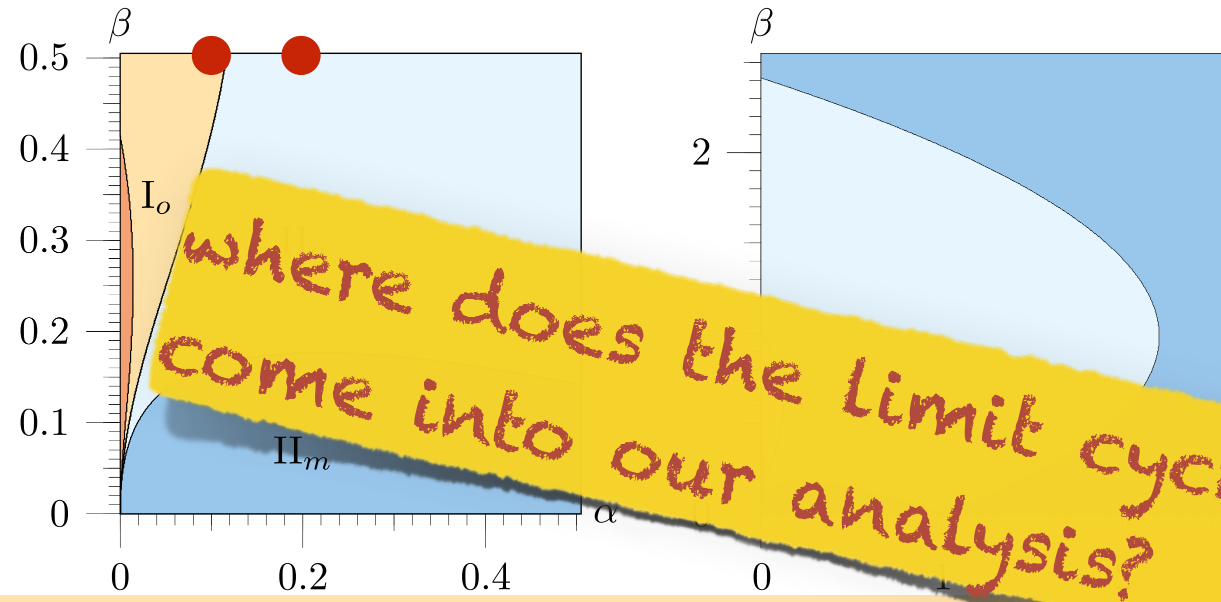


calculate det & tr

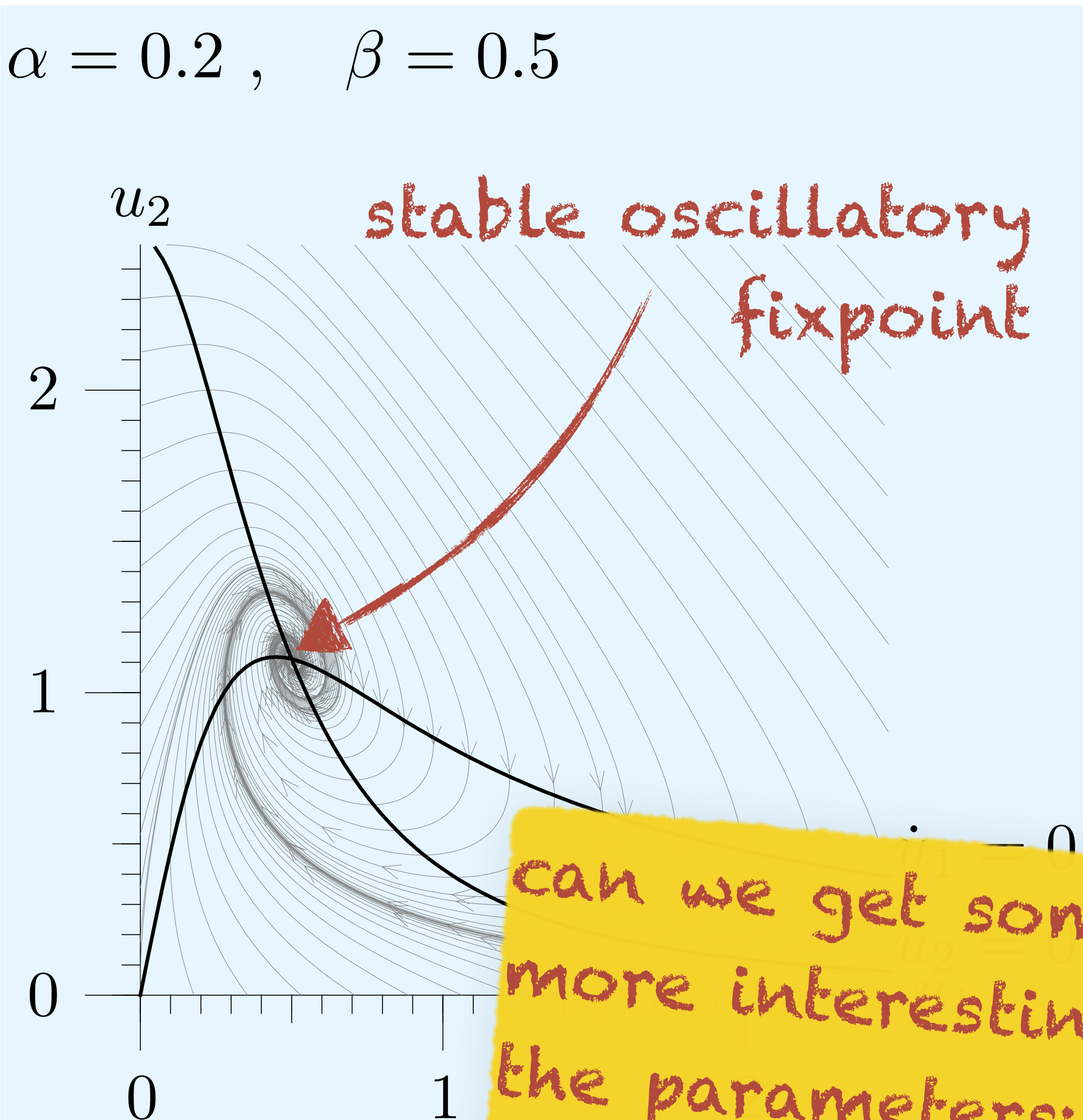
$$\det \mathsf{a} = \alpha + \beta^2 > 0$$

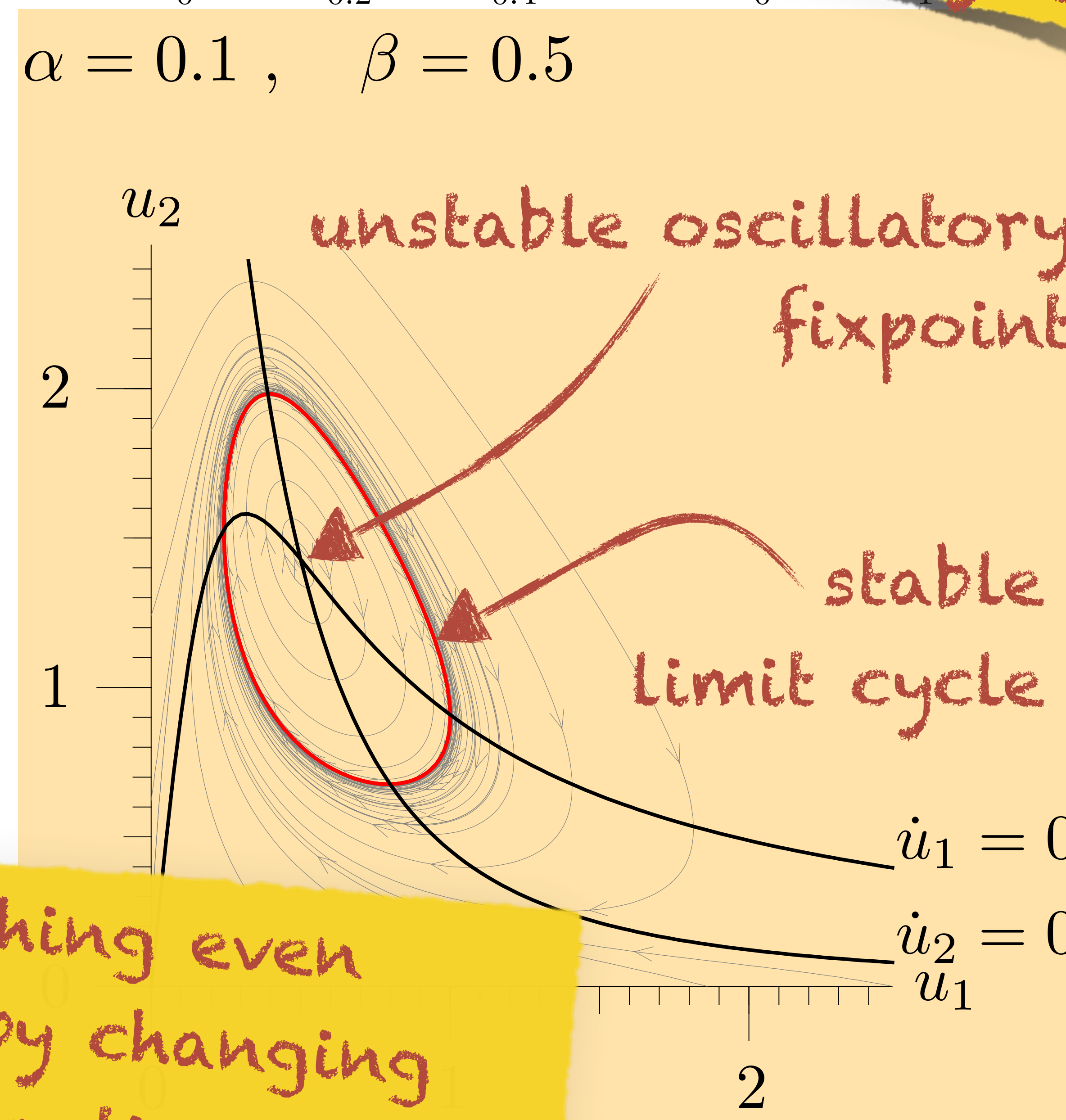
$$\mathrm{tr}\ \mathsf{a} = 1 - \alpha - \beta^2 - \frac{2\alpha}{\alpha + \beta^2}$$

phenomenology



where does the limit cycle come into our analysis?





can we get something even more interesting by changing the parameters: chaotic motion? any guess?